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Fuzzy Bilinear Functional

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Abstract

In this paper we introduce the concept of fuzzy bilinear functional in a vector space and we show that the resulting norm satisfies the Cauchy-Schwarz inequality.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [11]. Since then it has flourished as a vigorous field of research in engineering, medical science, social science, computer science and many other disciplines. Fuzzy fields and fuzzy linear spaces were defined by Nanda [9], and fuzzy vector spaces by Kastsaras and Liu [7]. Felbin [4] introduced the concept of fuzzy norm and showed that every finite dimensional fuzzy normed linear space has a completion. Xiao and Zhu [10] modified the definition of fuzzy norm and studied the topological properties of fuzzy normed linear spaces. For further references on the above or similar researches see also [1, 2, 3, 5, 6, 8].

In this work we introduce the concept of a fuzzy bilinear functional and we show that the resulting norm satisfies the Cauchy-Schwarz inequality.

2. Preliminaries

In this section, we recall the following definitions which are useful in the sequel:

Definition 1: Let $\mu,\gamma \in \mathbb{F}(E)$ and $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\mu \le \gamma$ if and only if $\mu_\alpha^- \le \gamma_\alpha^-$ and $\mu_\alpha^+ \le \gamma_\alpha^+$, for all $\alpha \in (0,1]$. Strict inequality in $\mathbb{F}(E)$ is defined by $\mu < \gamma$ if and only if $\mu_\alpha^- < \gamma_\alpha^-$ and $\mu_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0,1]$.

Definition 2: For a positive fuzzy real number μ we define $\sqrt{\mu} = \gamma$, where

$$[\gamma]_\alpha = \left[\sqrt{\mu_\alpha^-}, \sqrt{\mu_\alpha^+}\right]$$

for $\alpha \in (0,1]$.

Definition 3: Let X be a vector space over E. A fuzzy bilinear functional on X is a mapping $\varphi : X \times X \to \mathbb{F}(E)$ such that for all vectors $x, y, z \in X$ and all $r \in E$,

I. $\varphi(x+y,z) = \varphi(x,z) \oplus \varphi(y,z)$.

II. $\varphi(rx,y) = r\varphi(x,y)$.

III. $\inf_{\alpha\in(0,1]} \varphi(x,x)_\alpha^- > 0$ if $x \ne 0$.

Definition 4: A fuzzy bilinear functional is symmetric if $\varphi(x,y) = \varphi(y,x)$. Also, a fuzzy bilinear functional is positive if $\varphi(x,x) \ge \tilde{0}$.

Definition 5: The norm for vector space E by using fuzzy bilinear functional define as follows

$$\|x\|_\varphi = \sqrt{\varphi(x,x)} \tag{1}$$

for all $x \in E$.

In order to derive our main results, we need the following proposition:

Proposition 6: Let $\mu,\gamma \in \mathbb{F}(E)$ and $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. Then

I. $[\mu \pm \gamma]_\alpha = \left[\mu_\alpha^- \pm \gamma_\alpha^-, \mu_\alpha^+ \pm \gamma_\alpha^+\right]$.

II. $[\mu \times \gamma]_\alpha = \left[\mu_\alpha^- \gamma_\alpha^-, \mu_\alpha^+ \gamma_\alpha^+\right]$ for $\mu, \gamma \in \mathbb{F}^+(E)$.

III. $\left[\tilde{1}/\mu\right]_\alpha = \left[1/\mu_\alpha^+, 1/\mu_\alpha^-\right]_\alpha$, if $\mu_\alpha^- > 0$.

IV. $\left[|\mu|\right]_\alpha = \left[\max\left(0, \mu_\alpha^-, \mu_\alpha^+\right), \max\left(\left|\mu_\alpha^-\right|, \left|\mu_\alpha^+\right|\right)\right]$.

Proof: See Lemma 2.1 of [6].

3. Main Theorem

Our main result is the following:

Theorem 7: A fuzzy vector space E together with its corresponding norm $\|.\|_\varphi$ satisfy the Schwarz inequality

$$\varphi(x, y) \le \|x\|_\varphi \|y\|_\varphi .$$

Proof: Let $x, y \neq 0$ and $\left[\varphi(x, y)\right]_\alpha = \left[\varphi(x, y)_\alpha^-, \varphi(x, y)_\alpha^+\right]$, $\alpha \in (0,1]$. Suppose that

$$m_\alpha = \max\left(0, \varphi(x, y)_\alpha^-, -\varphi(x, y)_\alpha^+\right)$$

and

$$m_\alpha^' = \max\left(\varphi(x, y)_\alpha^-, \varphi(x, y)_\alpha^+\right).$$

Then, by Proposition 6,

$$\left[\varphi(x, y)\right]_\alpha = \left[m_\alpha, m_\alpha^'\right], \quad \alpha \in (0,1].$$

Let

$$\left[\varphi(x, x)\right]_\alpha = \left[\varphi(x, x)_\alpha^-, \varphi(x, x)_\alpha^+\right]$$

and

$$\left[\varphi(y, y)\right]_\alpha = \left[\varphi(y, y)_\alpha^-, \varphi(y, y)_\alpha^+\right],$$

for $\alpha \in (0,1]$. By Definition 2,

$$\left[\|x\|_{\varphi}\right]_{\alpha} = \left[\sqrt{\varphi(x,x)_{\alpha}^{-}}, \sqrt{\varphi(x,x)_{\alpha}^{+}}\right]$$

and

$$\left[\|y\|_{\varphi}\right]_{\alpha} = \left[\sqrt{\varphi(y,y)_{\alpha}^{-}}, \sqrt{\varphi(y,y)_{\alpha}^{+}}\right],$$

for $\alpha \in (0,1]$. We show that

$$m_{\alpha} = \left[\sqrt{\varphi(x,x)_{\alpha}^{-}}, \sqrt{\varphi(y,y)_{\alpha}^{-}}\right].$$

By Definition3, we deduce

$$\tilde{0} \le \varphi(x+ry, x+ry) = \varphi(x,x) + 2r\varphi(x,y) + r^{2}\varphi(y,y) \tag{2}$$

then

$$\tilde{0} \le \varphi(x+ry, x+ry)_{\alpha}^{-} = \begin{cases} \varphi(x,x)_{\alpha}^{-} + 2r\varphi(x,y)_{\alpha}^{-} + r^{2}\varphi(y,y)_{\alpha}^{-}, & r \ge 0 \\ \varphi(x,x)_{\alpha}^{-} + 2r\varphi(x,y)_{\alpha}^{+} + r^{2}\varphi(y,y)_{\alpha}^{-}, & r < 0 \end{cases}. \tag{3}$$

Case 1. Assume $\varphi(x,y)_{\alpha}^{+} < 0$. Let $r < 0$. The following condition are equivalent:

I. $2r\varphi(x,y)_{\alpha}^{+} \le -2rm_{\alpha}$;

II. $\left(2r\varphi(x,y)_{\alpha}^{+} + m_{\alpha}\right) \le 0$;

III. $\varphi(x,y)_{\alpha}^{+} + m_{\alpha} \ge 0.$

Since

$$m_{\alpha} = \max\left(0, -\varphi(x,y)_{\alpha}^{+}, \varphi(x,y)_{\alpha}^{-}\right) = -\varphi(x,y)_{\alpha}^{+}$$

it follows that

$$\varphi(x,y)_{\alpha}^{+} + m_{\alpha} = \varphi(x,y)_{\alpha}^{+} - \varphi(x,y)_{\alpha}^{+} \ge 0$$

and hence

$$2r\varphi(x,y)_{\alpha}^{+} \le -2rm_{\alpha}, \tag{4}$$

for each $r < 0$.

Next, let $r \ge 0$. The following condition are equivalent:

I. $2r\varphi(x,y)_{\alpha}^{-} \le -2rm_{\alpha}$;

II. $\left(2r\varphi(x,y)_\alpha^- + m_\alpha\right) \le 0;$

III. $\varphi(x,y)_\alpha^- + m_\alpha \le 0;$

IV. $\varphi(x,y)_\alpha^- - \varphi(x,y)_\alpha^+ \le 0.$

Hence

$$2r\varphi(x,y)_\alpha^- \le -2rm_\alpha, \qquad (5)$$

for each $r \ge 0$. Then by (3), (4) and (5), we obtain

$$0 \le \varphi(x,x)_\alpha^- - 2rm_\alpha + r^2\varphi(y,y)_\alpha^-,$$

for each $r \in E$.

Let $r = m_\alpha / \varphi(y,y)_\alpha^- \in E$. Since $\varphi(y,y)_\alpha^- \ge \inf_{\alpha\in(0,1]} \varphi(y,y)_\alpha^- > 0$, r is well defined and thus $m_\alpha^2 \le \varphi(x,x)_\alpha^- \varphi(y,y)_\alpha^-$. Also since $m_\alpha \ge 0$,

$$m_\alpha \le \sqrt{\varphi(x,x)_\alpha^-}\sqrt{\varphi(y,y)_\alpha^-}.$$

Case 2: Assume $\varphi(x,y)_\alpha^+ > 0$. Let $r < 0$. The following conditions are equivalent:

I. $2r\varphi(x,y)_\alpha^+ \le 2rm_\alpha;$

II. $2r\left(\varphi(x,y)_\alpha^+ - m_\alpha\right) \le 0;$

III. $\varphi(x,y)_\alpha^+ - m_\alpha \ge 0.$

Since $m_\alpha = \max\left(0, -\varphi(x,y)_\alpha^+, \varphi(x,y)_\alpha^-\right) = \max\left(0, \varphi(x,y)_\alpha^-\right),$

$$2r\varphi(x,y)_\alpha^+ \le 2rm_\alpha, \quad (r < 0). \qquad (6)$$

Next, let $r \ge 0$. The following conditions are equivalent:

I. $2r\varphi(x,y)_\alpha^- \le 2rm_\alpha;$

II. $\varphi(x,y)_\alpha^- \le m_\alpha.$

Hence

$$2r\varphi(x,y)_\alpha^- \le 2rm_\alpha, \quad \forall r \ge 0. \qquad (7)$$

Then by (3), (6) and (7), we have

$$2r\varphi(x,y)_\alpha^- + 2rm_\alpha + r^2\varphi(y,y)_\alpha^-, \quad (r \in E).$$

Let $r = {-m_\alpha}\big/{\varphi(y,y)_\alpha^-}$. Since $\varphi(y,y)_\alpha^- \ge \inf_{\alpha\in(0,1]}\varphi(y,y)_\alpha^- > 0$, r is well defined and thus

$$m_\alpha^2 \le \varphi(x,x)_\alpha^- \varphi(y,y)_\alpha^-.$$

Also since $m_\alpha \ge 0$,

$$m_\alpha \le \sqrt{\varphi(x,x)_\alpha^-}\sqrt{\varphi(y,y)_\alpha^-}.$$

Now, we show that $m_\alpha^{'} \le \sqrt{\varphi(x,x)_\alpha^+}\sqrt{\varphi(y,y)_\alpha^+}$. By (2),

$$0 \le \varphi(x+ry,x+ry)_\alpha^+ = \begin{cases} \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^+ + r^2\varphi(y,y)_\alpha^+, & r \ge 0 \\ \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^+ + r^2\varphi(y,y)_\alpha^+, & r < 0 \end{cases}. \tag{8}$$

From (3), we have

$$\begin{cases} 0 \le \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^- + r^2\varphi(y,y)_\alpha^+, & r \ge 0 \\ 0 \le \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^+ + r^2\varphi(y,y)_\alpha^+, & r < 0 \end{cases}. \tag{9}$$

Then, by (8) and (9),

$$0 \le \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^- + r^2\varphi(y,y)_\alpha^+, \quad (r \in E).$$

Let $r = -\varphi(x,y)_\alpha^- \big/ \varphi(y,y)_\alpha^+$. Then

$$\left(\varphi(x,y)_\alpha^-\right)^2 \le \varphi(x,x)_\alpha^+ \varphi(y,y)_\alpha^+$$

and hence

$$\varphi(x,y)_\alpha^- \le \sqrt{\varphi(x,x)_\alpha^+}\sqrt{\varphi(y,y)_\alpha^+}. \tag{10}$$

By (8) and (9), we have

$$0 \le \varphi(x,x)_\alpha^+ + 2r\varphi(x,y)_\alpha^+ + r^2\varphi(y,y)_\alpha^+, \quad (r \in E).$$

Let $r = -\varphi(x,y)_\alpha^+ \big/ \varphi(y,y)_\alpha^+$. Then

$$\left(\varphi(x,y)_\alpha^+\right)^2 \le \varphi(x,x)_\alpha^+ \varphi(y,y)_\alpha^+$$

and hence

$$\varphi(x,y)_{\alpha}^{+} \leq \sqrt{\varphi(x,x)_{\alpha}^{+}}\sqrt{\varphi(y,y)_{\alpha}^{+}}. \tag{11}$$

Now, by (10) and (11),

$$m_{\alpha}^{'} \leq \sqrt{\varphi(x,x)_{\alpha}^{+}}\sqrt{\varphi(y,y)_{\alpha}^{+}}.$$

Hence, by Definition1 and Proposition 6 we obtain

$$|\varphi(x,y)| \leq \|x\|_{\varphi}\|y\|_{\varphi}.$$

If $y=0$, then $\varphi(x,0)=\varphi(x,0)+\varphi(x,0)$ and hence

$$\varphi(x,0)_{\alpha}^{-} = \varphi(x,0)_{\alpha}^{-} + \varphi(x,0)_{\alpha}^{-}.$$

Thus, $\varphi(x,0)_{\alpha}^{-}=0$, for all $\alpha \in (0,1]$. Similarly $\varphi(x,0)_{\alpha}^{+}=0$. Consequently, $\varphi(x,y)=\varphi(x,0)=\tilde{0}$, which implies that

$$|\varphi(x,y)| = \tilde{0} \leq \tilde{0} = \|x\|_{\varphi}\|y\|_{\varphi}.$$

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