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Some Aspects on 2-Fuzzy Metric Spaces

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Abstract

In this paper the notion of 2-fuzzy metric space is introduced. Open balls and closed balls in 2-fuzzy metric spaces are also defined and various other concepts like fuzzy continuity, limit points, isolated points, convergent sequence and Cauchy sequence and several theorems are proved related to these concepts.

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1. Introduction

In 1965 Zadeh **[5]** introduced the concept of fuzzy sets. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. The concept of fuzzy metric space is one of such progressive development in the field of fuzzy topology. This has been investigated by many authors in different point and view. George and Veeramani **[1]** have introduced and studied the notion of fuzzy metric space and they proved that every metric induces a fuzzy metric. Grabiec**[2]** 1988 discussed fixed points in fuzzy metric space. Sharma and Kumar **[3]** introduced fuzzy 2-metric spaces, while Ha et al. **[4]** proved that a 2-metric need not be a continuous function.

In the present paper we define 2-fuzzy metric spaces over the set of all fuzzy sets on the universal set X. We also define open balls, convergent sequences, Cauchy sequences and continuous function. Various interesting theorems related to these concepts are established.

2. Definitions and Notations

Definition 2.1: A fuzzy set in X is a map from X to [0, 1].

Definition 2.2: Let F(X) = $[0,1]^{x}$ denote the set of all fuzzy sets on X. A 2- fuzzy set on X is a fuzzy set on F(X)

Definition 2.3: A 3 - triple (X , M , * *)* is said to be a fuzzy metric space if X is an arbitrary set, * is continuous t - norm and M is a fuzzy set on X x X x $(0,\infty)$ satisfying for all x, y, z $\in$ X and s , t > 0 the following conditions:

(i) M (x , y, t) > 0 .

(ii) M (x , y, t) = 1, if, and only if, x = y.

(iii) M (x , y , t) = M (y , x , t).

(iv) M (x , y , t) * M (y , z , s) $\leq$ M (x, z , t + s).

(v) M (x, y, t) : $(0, \infty) \rightarrow (0,1]$ is continuous.

Then M is called a fuzzy metric on X.

Definition 2.4: Let X be a linear space over the real field. A fuzzy subset N of X x R, (R is the set of all real numbers) is called a fuzzy norm on X, if the following conditions hold:

(N1) N(x , t) = 0, for all t $\in$ R with t $\leq$0.

(N2) For t $\in$ R with t > 0, N(x , t) = 1, if and only if, x = 0.

(N3) (N4) For all t $\in$ R with t > 0 N(cx , t) = N(x, $\frac{t}{c}$), if c $\neq$0 (c $\in$ R).

(N4) N(x + y, s + t) $\geq$ min {N(x, s) , N(y, t), for all s, t $\in$ R.

(N6) N(x, t) is a non decreasing function of R and $\lim_{t\to\infty}$ N(x, t) = 1.

Then the pair (X, N) is said to be a fuzzy normed linear space or in short FNLS.

3. 2 Fuzzy Metric Spaces

Definition3.1: Let $F(X)$ be a linear space over the real field. A fuzzy subset M of $F(X) \times F(X) \times R$, ($R$ the set of real numbers) is called a 2-fuzzy metric on X, if:

$(M1)$ $M(f_1, f_2, t) = 0$, for all $t \in R$ with $t \leq 0$,.

$(M2)$ For all $t \in R$ with $t > 0, M(f_1, f_2, t) = 1$ if and only if $f = g$.

$(M3)$ $M(f_1, f_2, t) = M(f_2, f_1, t)$.

$(M4)$ $M(f_1, f_2, t) * M(f_2, f_3, s) \leq M(f_1, f_3, t + s)$.

$(M5)$ $M(f_1, f_2, \cdot):(0, \infty) \to [0,1]$ is continuous.

Then $(F(X), M)$ is a fuzzy metric space and (X, M) is a 2- fuzzy metric space.

Example3.2 Let $F(X)$ be the set of all continuous functions from X to $[0,1]$ with $d(f(x), g(x)) = \sup_{x \in X} | f(x) - g(x)|$ and $a * b = ab, \forall\, a, b \in [0,1]$

Define $M(f, g, t) = \frac{t}{t+d(f,g)}$ .Then $(F(X), M,*)$ is a fuzzy metric space.

The function M defined above satisfies the following conditions:

(i) $M(f, g, t) = 1$

$\Leftrightarrow \frac{t}{t+d(f,g)} = 1$

$\Leftrightarrow d(f, g) = 0$

$\Leftrightarrow f = g.$

(ii) $M(f,g,t) > 0$ since $d(f,g) > 0$ as d is a metric.

(iii) $M(f,g,t) = \frac{t}{t+d(f,g)}$

$= \frac{t}{t+d(g,f)}$

$= M(g,f,t)$.

(iv) $M(f,g,t+s) = \frac{(t+s)}{(t+s)+d(f,g)}$

As $d(f,g) \leq d(f,g) + d(h,g)$.

$M(f,g,s+t) = \frac{s+t}{s+t+d(f,g)}$

$$\geq \frac{s+t}{s+t+d(f,h)+d(h,g)} \dots\dots\dots\dots\dots\dots\dots\dots.. (*)$$

$$\frac{s}{s+d(f,h)} > \frac{t}{t+d(h,g)} \Rightarrow \frac{s}{s+d(f,h)} - \frac{t}{t+d(h,g)} \geq 0$$

$$\Rightarrow s(t+d(h,g)) - t(s+(d(f,h))) \geq 0$$

$$\Rightarrow st + sd(h,g) - ts - td(f,h) \geq 0$$

$$\Rightarrow sd(h,g) - td(f,h) \geq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$\frac{s+t}{s+t+d(f,h)+d(h,g)} - \frac{t}{t+d(h,g)}$$

$$= \frac{(s+t)[t+d(h,g)] - t[s+t+d(f,h)+d(h,g)]}{[s+t+d(f,h)+d(h,g)][t+d(h,g)]}$$

$$= \frac{st + sd(h,g) + t^2 + td(h,g) - ts - t^2 - td(f,h) - td(h,g)}{[s+t+d(f,h)+d(h,g)][t+d(h,g)]}$$

$$= \frac{sd(h,g) - td(h,g)}{[s+t+d(f,h)+d(h,g)][t+d(h,g)]} \geq 0 \qquad \text{by (1)}$$

Therefore $\dfrac{s+t}{s+t+d(f,h)+d(h,g)} \geq \dfrac{t}{t+d(h,g)}$

Similarly if $\frac{t}{t+d(h,g)} \geq \frac{s}{s+d(f,h)}$

then $\dfrac{s+t}{s+t+d(f,h)+d(h,g)} \geq \dfrac{s}{s+d(f,h)}$

Thus from $(*)$

$$M(f,g,s+t) \geq \min\left\{\frac{s}{s+d(f,h)}, \frac{t}{t+d(h,g)}\right\}$$

Hence $M(f,g,s+t) \geq M(f,h,s) * M(h,g,t)$.

Definition3.3: For $t > 0$, the open ball $B(f,r,t)$ with center $f \in F(X)$ and radius $'r'$ $(0 < r < 1)$ is defined by

$$B(f,r,t) = \{g \in F(X): M(f,g,t) > 1 - r\}.$$

The closed ball $\bar{B}(f,r,t)$ with center $f \in F(X)$ and radius r $(0 < r < 1)$ is defined by

$$\bar{B}(f,r,t) = \{g \in F(X): M(f,g,t) \geq 1 - r\}.$$

Theorem3.4: Let $(X, M, *)$ be a 2-fuzzy metric space. The collection of all balls $B(f,r,t)$ for $f \in F(X)$ and $0 < r < 1$ is a base for the topology on $F(X)$. This topology is called the fuzzy metric topology on $F(X)$ induced by M and is denoted by τ_M.

Proof: Let $\mathcal{B} = \{B(f,r,t): f \in F(X); 0 < r < 1, t > 0\}$.

If $f \in F(X)$, then $f \in B(f,r,t)$ for some $0 < r < 1$.

$$\text{Hence } F(X) = \bigcup_{f \in F(X)} B(f,r,t)$$

Let $B(f_1,r_1,t_1)$ and $B(f_2,r_2,t_2)$ be any two members of $\mathcal{B}$.

If $\in B(f_1,r_1,t_1) \cap B(f_2,r_2,t_2)$, then $M(f_1,g,t_1) > 1 - r_1$ and $M(f_2,g,t_2) > 1 - r_2$.

Choose $0 < r_1 < r_2 < 1$ then $1 - r_2 < 1 - r_1$.

It follows that $M(f_1,g,t) > 1 - r_1 > 1 - r_2$ and if $0 < r_2 < r_1 < 1$ then $1 - r_1 < 1 - r_2$

It follows that $M(f_2,g,t) > 1 - r_2 > 1 - r_1$

Then either $B(f_1,r_1,t_1) \subseteq B(f_2,r_2,t_2)$ or $B(f_2,r_2,t_2) \subseteq B(f_1,r_1,t_1)$

Since $M(f_1,g_1,t_1) > 1 - r_1$, we can find a t_0, $0 < t_0 < 1$ such that $M(f_1,g_1,t_0) > 1 - r_1$

Let $r_0 = M(f_1,g_{1,}t_0) > 1 - r_1$

Since $r_0 > 1 - r_1$ we can find an $s, 0 < s < 1$ such that $r_0 > 1 - s > 1 - r_1$,

For given r_0 and s such that $r_0 > 1 - s$ we can find r_1' , $0 < r_1' < 1$ such that $r_0 * r_1' \geq 1 - s$

Consider the ball $B(g, 1 - r_1', \; t_1 - t_0)$.

We claim that $B(g, 1 - r_1', \; t_1 - t_0) \subset B(f_1,r_1,t_1)$

Now $h \in B(g, 1 - r_1', \; t_1 - t_0)$ implies that $M(g,h,t_1 - t_0) > 1 - (1 - r_1') = r_1'$

Therefore $M(f_1,h,t_1) \geq M(f_1,g,t_0) * M(g,h,t_1 - t_0) \geq r_0 * r_1' \geq 1 - s > 1 - r$

which implies that $h \in B(f_1,r_1,t_1)$

By a similar argument it turns out that $B(g, 1 - r_2', t_2 - t_0') \subset B(f_2,r_2,t)$

Definition3.5: A subset A of the fuzzy metric space $(F(X), M,*)$ is said to be open if given any $f \in A$, there exists $0 < r < 1$ and $t > 0$ such that $B(f, r, t) \subseteq A$.

Theorem3.6: Every open ball in the 2-fuzzy metric space is open.

Proof: Consider the open ball $B(f, r, t)$ and let $g \in B(f, r, t)$ which implies $M(f, g, t) > 1 - r$

Then there exists a t_0, $0 < t_0 < t$, such that $M(f, g, t_0) > 1 - r$

Let $r_0 = M(f, g, t_0) > 1 - r$

Again since $r_0 > 1 - r$, we can find , $0 < s < 1$ such that $r_0 > 1 - s > 1 - r$

and $r_0 * r_1 \geq 1 - s$. It is obvious that $B(g, 1 - r, \ t - t_0) \subset B(f, r, t)$

If $h \in B(g, 1 - r, \ t - t_0)$ then $M(g, h, t - t_0) > r_1$

Therefore $M(f, h, t) \geq M(f, g, t_0) * M(g, h, t - t_0) \geq r_0 * r_1 \geq 1 - s > 1 - r$

Hence $h \in B(f, r, t)$ and so $B(f, r, t)$ is open

Definition3.7: Let $(F(X), M)$ be a fuzzy metric space then $f \in F(X)$ is said to be an adherent point of a subset A of $F(X)$ if every open ball $B(f, r, t)$ with $0 < r < 1, t \in \mathbb{R}$. contains atleast one element of A. The adherent points are of two types

(i) Limit points or accumulation points
(ii) Isolated points

Definition3.8: Let $(F(X), M)$ be a fuzzy metric space . An element $f \in F(X)$ is said to be a limit point or accumulation point of a subset A of $F(X)$ if every open ball $B(f, r, t)$ with $0 < r < 1, t \in \mathbb{R}$. about f contains at least one element of A other than f.

Definition3.9: An adherent point f of $F(X)$ is called an isolated point if there exists atleast open ball $B(f, r, t)$ with centre f which contains no elements of A.

Definition3.10: Let $(F(X), M,*)$ be a fuzzy metric space then

(1) A sequence $\{f_n\}$ in $F(X)$ is said to converge to f in $F(X)$ if for each $\epsilon \in (0,1)$ and each $t \in R^+$, there exists $N \in z^+$ such that $M(f_n, f, t) > 1-\epsilon$ for all $n \geq N$ or $\lim_{n\to\infty} M(f_n, f, t) = 1$.

(2) A sequence $\{f_n\}$ in $F(X)$ is said to be Cauchy if for each $\epsilon \in (0,1)$ and each $t \in R^+$, there exists $N \in z^+$ such that $M(f_n, f_m, t) > 1-\epsilon$ for all $m, n \geq N$ or $M(f_n, f_m, t) = 1$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Theorem3.11: Let $\{f_n\}, \{g_n\}$ be sequence in fuzzy metric space $F(X)$ and for all $r_1 \in (0,1)$ there exist $r \in (0,1)$ such that $r * r \geq r_1$ then every convergent sequence has a unique limit.

Suppose $f_n \to f$ and $f_n \to g$ and $f \neq g$ then if for ϵ_1 and $\epsilon_2 \in (0,1)$ and $s, t > 0$ there exists $N_1, N_2 \in Z_+$, such that $M(f_n, f, s) > 1 - \epsilon_1$, for all $n \geq N_2$,

$M(f_n, g, t) > 1 - \epsilon_2$, for all $n \geq N_2$

Then $M(f, g, s+t) \geq M(f_n, f, s) * M(f_n, g, t) > (1-\epsilon_1) * (1-\epsilon_2)$

$> 1 - \epsilon$ for all $n \geq N$ where $N = \max\{N_1, N_2\}$. Therefore $f = g$.

Definition3.12: Let $F(X)$ be a linear space over a field K. A fuzzy subset N of $F(X) \times \mathbb{R}$ is called a fuzzy norm on $F(X)$ if and only if $\forall f, g \in F(X)$ and $c \in K$.

(N1) $\forall t \in \mathbb{R}$ with $t \leq 0, N(f,t) = 0$.

(N2) $\forall t \in \mathbb{R}$ with $t > 0, N(f,t) = 1$ if and only if $f = \underline{0}$.

(N3) $\forall t \in \mathbb{R}$ with $t > 0, N(cf, t) = N\left(f, \frac{t}{|c|}\right)$ if $c \neq 0$.

(N4) $\forall\, s,t \in \mathbb{R}, f,g \in F(X), N(f+g,s+t) \geq \min\{N(f,s),N(g,t)\}$

(N5) $N(f,\ \cdot)$ is a non decreasing function of $\mathbb{R}$ and $\lim_{n\to\infty} N(f,t) = 1$.

The pair $(F(X),N)$ is referred to a fuzzy normed linear space or (X,N) is known as the 2-fuzzy normed linear space.

Theorem3.13: Given a 2-fuzzy normed linear space, define a two fuzzy metric M on X as

$$M(f,g,t) = N(f-g,t) \text{ for all } f,g \in F(X), t \in \mathbb{R}.$$

Proof: Given $M(f,g,t) = N(f-g,t)$ for all $f,g \in F(X)$, $t \in \mathbb{R}$.

(M1) $\forall\, t \in \mathbb{R}$, with $t \leq 0, M(f,g,t) = N(f-g,t) = 0$ by (N1)

(M2) $\forall\, t \in \mathbb{R}$, with $t > 0, M(f,g,t) = N(f-g,t) = 1$ iff $f-g=0 \Leftrightarrow f=g$.

(M3) $M(f,g,t) = N(f-g,t) = N\left(g-f,\frac{t}{|-1|}\right) = N(g-f,t) = M(g,f,t)$

(M4) $M(f,g,s+t) = N(f-g,s+t) = N(f-h+h-g,s+t)$

$$\geq N(f-h,s) * N(h-g,t) = M(f,h,s) * M(h,g,t)$$

(M5) $M(f,g,\ \cdot)$ is a continuous non decreasing function of $\mathbb{R}$ and $\lim_{n\to\infty} N(f,g,t) = 1$, because $M(f,\ \cdot)$ is continuous non decreasing function and $\lim_{n\to\infty} N(f,t) = 1$.

Theorem 3.14: (i) If $\{f_n\}$ convergence to f then $\{kf_n\}$ converges to kf for $k \in K-\{0\}$ and K is a field.

(ii) If f_n converges to f and g_n converges to f then $f_n + g_n$ converges to $f+g$.

Proof: (i) Given $f_n \to f$ for given $\epsilon(\in (0,1))$ and $t > 0$ there exist $N \in Z_+$ such that $M(f_n, f, t) > 1-\epsilon \Rightarrow N(f_n - f, t) > 1-\epsilon$

Then $N(kf_n - kf, t)\ N(k(f_n - f), t) = N\left(f_n - f, \frac{t}{|k|}\right) > 1 - \epsilon$

$\Longrightarrow M(kf_n, kf, t) = N(kf_n - kf, t) > 1 - \epsilon$

So $kf_n \to kf$ for $k \in K - \{0\}$ and K is a field.

(ii) If $f_n \to f$ and $g_n \to g$ then for given $\epsilon_1, \epsilon_2\ (\epsilon(0,1))$ and $t_1, t_2 > 0$ there exists N_1, N_2 such that $M(f_n, f, t_1) > 1 - \epsilon_1, \forall\, n \geq N_1$

$$\Rightarrow N(f_n - f, t_1) > 1 - \epsilon_1, \forall\, n \geq N_1$$

and $M(g_n, g, t_1) > 1 - \epsilon_2, \forall\, n \geq N_2$

$$\Rightarrow N(g_n - g, t_2) > 1 - \epsilon_2, \forall\, n \geq N_2$$

Consider $M(f_n + g_n, f + g, t_1 + t_2) = N(f_n + g_n, f - g, t_1 + t_2)$

$$= N(\overline{f_n - f} + g_n - g, t_1 + t_2)$$

$\geq N(f_n - f, t_1) * N(g_n - g, t_2)$

$\geq (1 - \epsilon_1) * (1 - \epsilon_2)$ $\quad$ ($\exists\, \epsilon(\epsilon(0,1))$ such that $(1 - \epsilon_1) * (1 - \epsilon_2) > 1 - \epsilon$)

$> 1 - \epsilon$, for all $n \geq N = \max\{N_1, N_2\}$.

Definition3.15: Let $(F(X), M_1, *)$ and $(F(Y), M_2, *)$ be fuzzy metric space. A function $T: F(X) \to F(Y)$ is fuzzy continuous. If for any open set V in $F(Y)$, let $f_0 \in T^{-1}(V)$ there exists $r, 0 < r < 1$ such that $B(g_0, r, t) \subset V$ where $T(f_0) = g_0$.

Theorem 3.16: Let $(F(X), M_1, *)$ and $(F(Y), M_2, *)$ be fuzzy metric space. Then a function $T: F(X) \to F(Y)$ is fuzzy continuous, if and only if, for each $f \in F(X)$ and $\epsilon\ (\epsilon \in (0,1))$ there exists $\delta\big(\epsilon(0,1)\big)$ such that $M_1(f, h, t) > 1 - \delta$ implies $M_2(Tf, Th, t) > 1 - \epsilon$.

Proof: Consider $B(f, \delta, t) = \{g \in F(X): M_1(f, g, t) > 1 - \delta\}$.

$$B'(f',\epsilon,t) = \{g' \in F(Y): M_2(f',g',t) > 1-\epsilon\}$$

Assume that T is continuous. For each $f \in F(X)$ and V an open set containing $T(f)$ there exists $r, 0 < r < 1$ such that $B(T(f),\epsilon,t) \subset V$.

For each open ball $B(T(f),\epsilon,t)$ there exists an open ball $B(f,\delta,t)$

such that $T(B(f,\delta,t)) \subset B(T(f),\epsilon,t)$

Therefore $h \in B(f,\delta,t) \Rightarrow T(h) \in B(T(f),\epsilon,t)$

$$M_1(f,h,t) > 1-\delta \Rightarrow M_2(T(f),T(h),t) > 1-\epsilon$$

Conversely suppose that the condition holds. Let $B(Tf,\epsilon,t)$ be an open ball containing $T(f)$, then there exists $\delta(\epsilon(0,1))$ satisfying the condition . Let $B(f,\delta,t)$ be an open ball around f. Then $h \in B(f,\delta,t) \Rightarrow M_1(f,h,t) > 1-\delta \Rightarrow M_2(Tf,Th,t) > 1-\epsilon \Rightarrow Th \in B(Tf,\epsilon,t)$, which implies that T is continuous.

Definition 3.17: Let $(F(X),M,*)$ be a fuzzy metric space and A be a non empty subset of $F(X)$. If $f \in F(X)$ then the distance of f to A is defined by f and A where $f \in F(X)$ is defined as $d(f,A) = \inf\{t: M(f,g,t) \geq 0, g \in A\}$

Theorem 3.18: An element f of the fuzzy metric space $(F(X),M,*)$ is an adherent point of the subset A of $F(X)$ if and only if $d(f,A) = 0$.

Proof: By definition $d(f,A) = \inf\{t: M(f,g,t) \geq 0, g \in A\}$

Therefore $d(f,A) = 0 \Rightarrow$ every open ball $B(f,r,t)$ with centre $\in F(X)$, $0 < r < 1$, $t \in R$ contains an element of $F(X)$. which implies that f is an adherent point of A.

Conversely if f is an adherent point of A then with f as an isolated point of A $d(f,A) = 0$.

Theorem 3.19: Every 2-fuzzy metric space is Hausdorff.

Proof: Let $(F(X), M,*)$ be the fuzzy metric space. Choose two distinct elements $f, g \in F(X)$ such that $0 < M(f, g, t) < 1$

Suppose $M(f, g, t) = r$ for $0 < r < 1$

For each r_0 with $r < r_0 < 1$, there exists r_1, such that $r_1 * r_1 \geq r_0$.

Consider the open balls $B\left(f, 1 - r_1, \frac{t}{2}\right)$ and $B\left(g, 1 - r_1, \frac{t}{2}\right)$ about f and g.

Then these two balls are the required disjoint neighborhoods containing f and g.

Suppose if $h \in B\left(f, 1 - r_1, \frac{t}{2}\right) \cap B\left(g, 1 - r_1, \frac{t}{2}\right)$,

then $M(f, g, t) \geq M\left(f, h, \frac{t}{2}\right) * M\left(g, h, \frac{t}{2}\right) \geq r_1 * r_1 \geq r_0 > r$, which is a contradiction, Therefore $F(X)$ is Hausdorff.

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