Fuzzy Logic and Uncertainty in Mathematics Education

Michael Gr. Voskoglou

Professor of Mathematical Sciences
Graduate Technological Educational Institute
School of Technological Applications- 26334 Patras, Greece
e-mail: voskoglou@teipat.gr, mvosk@hol.gr

In memoriam of dear friend and colleague Filippo Spagnolo

Abstract

We develop a general model for representing several processes in Mathematics Education (e.g. learning, mathematical modelling, problem-solving, etc) involving fuzziness and uncertainty. To each of the main stages of these processes we correspond a fuzzy subset of the set of the linguistic labels of negligible, low intermediate, high and complete success respectively by students at this stage and we use the total possibilistic uncertainty as a measure of students’ capacities. Two classroom experiments are also presented for the problem-solving case illustrating the use of our model in practice. Similar models were developed in earlier papers in the areas of Artificial Intelligence (Case-Based Reasoning) and Management.

Keywords: Fuzzy sets and relations, Learning, Mathematical modeling, Measures of uncertainty, Possibility theory, Problem-solving.

1 Introduction

They appear often didactic situations in Mathematics Education characterized by a degree of fuzziness and/or uncertainty (e.g. learning, mathematical modelling, problem-solving, etc). In fact, students’ cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from teacher’s point of view there usually exists vagueness about the degree of success of students in each of the stages of the corresponding didactic situation. All these gave us the impulsion to introduce principles of fuzzy logic and of uncertainty theory in an effort to describe in a more effective way the process of such kind of situations in classroom. Therefore our target in this paper is to construct a general (fuzzy) model that could be adapted in each particular case in order to represent the process of the corresponding didactic situation.
The concept of uncertainty, which emerges naturally within the broad framework of fuzzy sets theory, is involved in any didactic situation, especially when dealing with real-world problems. Uncertainty is a result of some information deficiency. In fact, information pertaining to the model within which a real situation is conceptualized may be incomplete, fragmentary, not full reliable, vague, contradictory, or deficient in some other way. Thus the amount of information obtained by an action can be measured in general by the reduction of uncertainty resulting from the action. In other words the amount of uncertainty regarding some situation represents the total amount of potential information in this situation.

For general facts on fuzzy sets and uncertainty theory we refer freely to the book of Klir and Folger [16].

2 The general model

Let us consider a group of \( n \) students, \( n \geq 2 \), in classroom. Denote by \( S_i, \ i = 1,2,3 \), the main stages of the process of the didactic situation that we want to represent and by \( a, b, c, d, \) and \( e \) the linguistic labels of negligible, low, intermediate, high and complete success respectively of a student in each of the \( S_i \)'s.

Set \( U = \{a, b, c, d, e\} \). We are going to attach to each stage \( S_i \) a fuzzy subset, \( A_i \) of \( U \).

For this, if \( n_{ia}, n_{ib}, n_{ic}, n_{id} \) and \( n_{ie} \) denote the number of students that faced negligible, low, intermediate, high and complete success at stage \( S_i \) respectively, \( i=1,2,3 \), we define the membership function \( m_{A_i} \) for each \( x \) in \( U \), as follows:

\[
m_{A_i}(x) = \begin{cases} 
1, & \text{if } \frac{4n}{5} < n_{ix} \leq n \\
0.75, & \text{if } \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\
0.5, & \text{if } \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\
0.25, & \text{if } \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\
0, & \text{if } 0 \leq n_{ix} \leq \frac{n}{5} 
\end{cases}
\]

Then the fuzzy subset \( A_i \) of \( U \) corresponding to \( S_i \) has the form:

\[
A_i = \{(x, m_{A_i}(x)): \ x \in U\}, \ i = 1, 2, 3.
\]

In order to represent all possible student profiles (overall states) during the corresponding process we consider a fuzzy relation, say \( R \), in \( U^3 \) of the form

\[
R= \{(s, m_R(s)): s=(x, y, z) \in U^3\}.
\]

We make the hypothesis that the stages of the process of the corresponding didactic situation are depended to each other. This means that the degree of success of a student in a certain stage depends upon the degree of his/her success in the previous stages, as it usually happens in practice. Under this hypothesis and in order to determine properly the membership function \( m_R \) we give the following definition:
Definition 2.1: A profile $s=(x, y, z)$, with $x, y, z$ in $U$, is said to be well ordered if $x$ corresponds to a degree of success equal or greater than $y$, and $y$ corresponds to a degree of success equal or greater than $z$.

For example, $(c, c, a)$ is a well ordered profile, while $(b, a, c)$ is not.

We define now the membership degree of a profile $s$ to be

$$m_R(s) = m_{A_x}(x)m_{A_y}(y)m_{A_z}(z),$$

if $s$ is well ordered, and $0$ otherwise.

In fact, if for example profile $(b, a, c)$ possessed a nonzero membership degree, how it could be possible for a student, who has failed during the middle stage, to perform satisfactorily in the next stage?

In the next, for reasons of brevity, we shall write $m_s$ instead of $m_R(s)$. Then the possibility $r_s$ of profile $s$ is defined by

$$r_s = \frac{m_s}{\max\{m_s\}},$$

where $\max\{m_s\}$ denotes the maximal value of $m_s$, for all $s$ in $U^3$. In other words the possibility of $s$ expresses the “relative membership degree” of $s$ with respect to $\max\{m_s\}$.

As we have seen above the amount of information obtained by an action can be measured by the reduction of uncertainty resulting from the action. Accordingly students’ uncertainty during the process of the corresponding didactic situation is connected to students’ capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of students’ capacities. Within the domain of possibility theory uncertainty consists of strife (or discord), which expresses conflicts among the various sets of alternatives, and non-specificity (or imprecision), which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([17]; p.28). Strife is measured by the function $ST(r)$ on the ordered possibility distribution $r$: $r_1 \geq r_2 \geq \ldots \geq r_n \geq r_{n+1}$ of the student group defined by

$$ST(r) = \frac{1}{\log 2} \left\{ \sum_{i=2}^{n} (r_i - r_{i+1}) \log i \right\} - \sum_{j=1}^{n} r_j,$$

while non-specificity is measured by the function

$$N(r) = \frac{1}{\log 2} \left\{ \sum_{i=2}^{n} (r_i - r_{i+1}) \log i \right\}.$$

The sum $T(r) = ST(r) + N(r)$ is a measure of the total possibilistic uncertainty for ordered possibility distributions. Therefore the total possibilistic uncertainty of the student group during the process can be adopted as a measure of students’ capacities. This is reinforced by Shacklere [34], who argues that human reasoning can be formalized more adequately by possibility theory rather, than by probability theory. The lower is the value of $T(r)$ (which means greater reduction of the initially existing uncertainty), the better the performance of the student group during the process of the corresponding didactic situation.

Assume finally that one wants to study the combined results of behaviour of $k$ different student groups, $k \geq 2$, during the same process. For this we introduce the fuzzy variables $A_1(t), A_2(t)$ and $A_3(t)$ with $t=1, 2, \ldots, k$. The values of these variables represent fuzzy subsets of $U$ corresponding to the stages of the process for each of the $k$ student groups; e.g. $A_1(2)$ represents the fuzzy subset of $U$ corresponding to the
stage of planning for the second group (t=2). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the possibility \( r(s) \) of each student profile \( s \) with respect to the membership degrees of \( s \) for all student groups. For this reason we introduce the pseudo-frequencies \( f(s) = \sum_{t=1}^{k} m_j(t) \) and we define \( r(s) = \frac{f(s)}{\max\{f(s)\}} \), where \( \max\{f(s)\} \) denotes the maximal pseudo-frequency. Obviously the same method could be applied when one wants to study the combined results of behaviour of a student group during \( k \) different didactic situations.

3 Past applications of the fuzzy model

In this section we shall sketch applications of the above model (or similar ones) presented in earlier papers for a more effective description of several situations involving fuzziness and uncertainty, mainly in the area of Mathematics Education, but also in the areas of Artificial Intelligence and Management.

3.1 The process of learning

The concept of learning is fundamental to the study of human cognitive action; but while everyone knows in general what learning is, the understanding of its nature has proved to be complicated. This basically happens because it is very difficult for someone to understand the way in which the human mind works, and therefore to describe the mechanisms of the acquisition of knowledge by the individual. The problem is getting even harder by taking into consideration the fact that these mechanisms, although they appear to have some common general characteristics, actually they differ in details from person to person.

Over the last four decades mathematics education has addressed philosophical and epistemological perspectives with respect to mathematics learning. It has become common to think of mathematics in fallibilistic terms [9, 12, 40], to consider learning as a constructive process [7, 46], to situate knowledge and learning relative to communities of practice [19] and to debate the commensurability of constructivist and sociocultural learning theories [22, 43]. Theoretical considerations like the nature of mathematical knowledge, what it means to know mathematics and to come to know it, how knowing in mathematics is related to knowing in social settings more widely, have been deeply considered and seriously debated [2, 5, 6, 15]. The mathematics education discipline has become mature in such theoretical considerations. Voss [66] adopted an argument raised much earlier by Ferguson [11] and others that learning is a specific case of the general class of transfer of knowledge and therefore any instance of learning involves the use of already existing knowledge. Accordingly learning basically consists of successive problem – solving activities, in which the input information is represented of existing knowledge, with the solution occurring when the input is appropriately interpreted. The process involves the following stages: Representation of the input data, interpretation of these data in order to produce the new knowledge, generalization of the new knowledge to a variety of situations and categorization of the generalized knowledge.

More explicitly the representation of the stimulus input is relied upon the individual’s ability to use contents of his (her) memory in order to find information that will facilitate a solution development. Learning consists of developing an appropriate
number of interpretations and generalizing them to a variety of situations. When the knowledge becomes substantial, much of the process involves categorization, i.e. the input information is interpreted in terms of the classes of the existing knowledge. Thus the individual becomes able to relate new information to his (her) knowledge structures that have been variously described as schemata, or scripts, or frames.

In developing our fuzzy model for the process of learning we considered a group of \( n \geq 2 \) students, during the learning process of a subject matter in classroom and we denoted by \( S_i, i=1, 2, 3 \), the stages of interpretation, generalization and categorization of the Voss’s model. To each of the \( S_i \)’s we attached a fuzzy subset \( A_i \) of \( U \) (see section 2) by defining the membership function \( m_{Ai} \) in terms of the frequencies, i.e. by

\[
m_{Ai}(x) = \frac{n_{ix}}{n} \quad \text{for each } x \in U.
\]

Thus we can write

\[
A_i = \{(x, \frac{n_{ix}}{n}) : x \in U\}.\]

The development of our fuzzy model for learning follows then the general lines presented in the previous section. For more details and classroom applications of the model see [49], [57] and [61].

Notice that in the same way as above we could also attach to the stage of representation of Voss’s model a fuzzy subset of \( U \). However this, although it makes technically the presentation of our fuzzy model much more complicated, is not so important, since representation, although it deserves some attention, is actually an introductory step of the process of learning. The above manipulation is a simplification made to the real system in order to transfer from it to the “assumed real system”. This is a standard technique applied during the modelling process of real world problems, which enables the formulation of the problems in a form ready for mathematical treatment [54; section 1].

In general, learning is a very complex process that takes place not only in the class, but also between classes, or after a school day is over, or even in unexpected moments (e.g. during sleep). Therefore, apart of making simplifications, it is inevitable to put some restrictions in order to obtain a mathematical description of the learning process. The basic restriction in our case is that we consider the process of learning a subject matter during the teaching process in the classroom only and not the process of learning by the individual in general. Under this restriction one must keep in mind that, as it frequently happens, a learner may not be able to pass successfully through all stages of the learning process in classroom. The step of categorization for example could be reached out of class, or in a next class, but for the particular chronological moment of our study this is counted as a student’s failure to reach categorization.

### 3.2 Mathematical modelling

Mathematical modelling appears today as a dynamic tool for teaching mathematics, because it connects mathematics with real world and our everyday life, thus giving students the opportunity to realize its usefulness in practical applications [52]. From the time that Pollak [28] represented mathematical modelling as a cyclic interaction between mathematics and real world in ICME-3, Karlsruhe, 1976, much effort has been placed by authors and researchers to analyze in detail its process [14, 26]. In all models developed it is acceptable in general (with minor variations) that the main stages of the mathematical modelling process involve:

- **Analysis** of the given real world problem, i.e. understanding the statement and recognizing limitations, restrictions and requirements of the real system.
• **Mathematization**, i.e. formulation of the real situation in such a way that it will be ready for mathematical treatment, and construction of the model.

• **Solution** of the model, achieved by proper mathematical manipulation.

• **Validation** (control) of the model, usually achieved by reproducing through it the behaviour of the real system under the conditions existing before the solution of the model (empirical results, special cases etc).

• **Implementation** of the final mathematical results to the real system, i.e. “translation” of the mathematical solution obtained in terms of the corresponding real situation in order to reach the solution of the given real problem.

As it is mentioned in [14], all the above models are helpful in understanding what might be termed “ideal behaviour”, in which modellers proceed effortlessly from a real world problem through a mathematical model to acceptable solutions and report on them. However life in the classroom is not like that. In fact, recent research reports that students in school when tackling problems of mathematical modelling take individual routes associated with their individual learning styles [3, 8, 13, etc]. Therefore, one of the main advantages of our fuzzy model representing mathematical modelling is that it gives a realistic view of what happens in classroom through the study of all possible students’ profiles during the modelling process.

Notice that, the analysis of the problem, although it deserves some attention as being a prerequisite to mathematization, could be considered as an introductory stage of the modelling process. Therefore in developing our fuzzy model for the description of the mathematical modelling process we considered fuzzy subsets of $U$ corresponding only to the main stages of mathematization, solution and validation-implementation (as a joined stage) of the mathematical modelling process and we proceeded following the lines of the general mode described in section 2. For more details and classroom applications of this model look at [63] and [65].

### 3.3 Case-Based Reasoning

Case-Based Reasoning (CBR) is a general paradigm for problem-solving and learning from expertise, which is not only a psychological theory of human cognition, but it also provides a foundation for a new technology of intelligent computer systems that can solve problems and adapt to new situations. Broadly construed CBR is the process of solving new problems based on the solutions of similar past problems. Its coupling to learning occurs as a natural by-product of problem solving. When a problem is successfully solved, the experience is retained in order to solve similar problems in future. When an attempt to solve a problem fails, the reason for the failure is identified and remembered in order to avoid the same mistake in future. Thus CBR is a cyclic and integrated process of solving a problem, learning from this experience, solving a new problem, etc. It must be noticed that the term problem-solving is used here in a wide sense, which means that it is not necessarily the finding of a concrete solution to an application problem, it may be any problem put forth by the user. For example, to justify or criticize a proposed solution, to interpret a problem situation, to generate a set of possible solutions, or generate explanations in observable data, are also problem solving situations.

A lawyer, who advocates a particular outcome in a trial based on legal precedents, or an auto mechanic, who fixes an engine by recalling another car that exhibited similar
symptoms, are using CBR; in other words CBR is a prominent kind of analogy making.

CBR traces its roots in Artificial Intelligence to the work of Roger Schank and his students at Yale University, U.S.A. in early 1980’s. Schank’s model of dynamic memory [35] was the basis of the earliest computer intelligent systems that can be viewed as prototypes for CBR systems, the Kolodner’s CYRUS [18] and Lebowitz’s IPP [20]. An alternative approach is the category and exemplar model applied first to the PROTOS system of Porter and Bareiss [33], while some other types of memory models, developed later on.

The CBR systems expertise is embodied in general in a collection (library) of past cases rather, than being encoded in classical rules. Each case typically contains a description of the problem plus a solution and/or the outcomes. The knowledge and reasoning process used by an expert to solve the problem is not recorded, but is implicit in the solution.

As an intelligent-systems method CBR has got a lot of attention over the last few years, because it enables the information managers to increase efficiency and reduce cost by substantially automating processes. CBR first appeared in commercial systems in the early 1990’s and since then has been sued to create numerous applications in a wide range of domains including diagnosis, help-desk, assessment, decision support, design, etc. Organizations as diverse as IBM, VISA International, Volkswagen, British Airways and NASA have already made use of CBR in applications such as customer support, quality assurance, aircraft maintenance, process planning and many more applications that are easily imaginable.

CBR has been formalized for purposes of computer and human reasoning as a four steps process. These steps involve:

R₁: **Retrieve** the most similar to the new problem past case.

R₂: **Reuse** the information and knowledge of the retrieved case for the solution of the new problem.

R₃: **Revise** the proposed solution.

R₄: **Retain** the part of this experience likely to be useful for future problem-solving. More specifically, the retrieve task starts with the description of the new problem, and ends when a best matching previous case has been found. The subtasks of the retrieving procedure involve: Identifying a set of relevant problem descriptors, matching the case and returning a set of sufficiently similar cases given a similarity threshold of some kind, and selecting the best case from the set of cases returned. Some systems retrieve cases based largely on superficial syntactic similarities among problem descriptors, while advanced systems use semantic similarities. The reuse of the solution of the retrieved case in the context of the new problem focuses on two aspects: The differences between the past and the current case, and what part of the retrieved case can be transferred to the new case. Usually in non trivial situations part of the solution of the retrieved case cannot be directly transferred to the new case, but requires an adaptation process that takes into account the above differences.
Through the revision the solution generated by reuse is tested for success – e.g. by being applied to the real world environment, or to a simulation of it, or evaluated by a specialist – and repaired, if failed. When a failure is encountered, the system can then get a reminding of a previous similar failure and use the failure case in order to improve its understanding of the present failure, and correct it. The revised task can then be retained directly (if the revision process assures its correctness), or it can be evaluated and repaired again.

The final step $R_4$ involves selecting which information from the new case to retain, in what form to retain it, how to index the case for better retrieval in future for similar problems, and how to integrate the new case in the memory structure. Slade [41; Figure 1], Lei et al ([21]; Figure 1), Aamodt and Plaza ([1]; Figures 1 and 2) and others have presented detailed flowcharts illustrating the basic steps of the CBR process, while in an earlier paper [64] we have also presented a detailed analysis of the CBR methodology.

As a general problem-solving methodology intended to cover a wide range of real-world applications, CBR must face the challenge to deal with uncertain, incomplete and vague information. In fact, uncertainty is already inherent in the basic CBR hypothesis demanding that similar problems have similar solutions. Correspondingly recent years have witnessed an increased interest in formalizing parts of the CBR methodology within different frameworks of reasoning under uncertainty, and in building hybrid approaches by combining CBR with methods of uncertain and approximate reasoning. Fuzzy sets theory can be mentioned as a particularly interesting example. In fact, even though both CBR and fuzzy systems are intended as cognitively more plausible approaches to reasoning and problem-solving, the two corresponding fields have emphasized different aspects that complement each other in a reasonable way. Thus fuzzy set-based concepts and methods can support various aspects of CBR including: Case and knowledge representation, acquisition and modeling, maintenance and management of CBR systems, case indexing and retrieval, similarity assessment and adaptation, instance-based and case-based learning, solution explanation and confidence, and representation of context. On the other way round ideas and techniques for CBR can contribute to fuzzy set-based approximate reasoning. Here we shall sketch our development of a fuzzy representation for CBR systems.

For this, we considered such a system whose library contains $n$ past cases, $n \geq 2$ and based on the history of the past cases at each step of the CBR process, we corresponded a fuzzy subset of $U$ to each of the steps of retrieval, reuse and revision of the CBR process exactly as it has been described in section 2. Notice that there is no need to include the step of retaining ($R_4$) in our fuzzy representation, because all the past cases, either successful, or not, are retained in the system’s library and therefore there is no fuzziness in this case. In other words, keeping the same notation as in section 2, we have that $n_{4a} = n_{4b} = n_{4c} = n_{4d} = 0$ and $n_{4e} = 1$. The resulting total possibilistic uncertainty of the CBR system under study is connected to its effectiveness in solving new related problems (great effectiveness means small uncertainty). Notice that in general, the more are the stored past cases in the system’s library, the greater is expected to be its effectiveness. In fact, the more are the past cases, the greater is the probability for a new related problem to fit satisfactorily to one of them. Therefore on comparing the behaviour of two different
CBR systems designed for the solution of the same category of problems, apart of the values of their total possibilistic uncertainty one must take also in mind the number of stored past cases in their libraries. For a detailed description of the fuzzy representation of a CBR system and relative examples see [59].

3.4 Management

In [51] we presented a similar fuzzy model for evaluating the data of market’s research about the consumers’ correspondence for the negotiable products of a company, the level of which is characterized by fuzzy linguistic labels.

4 A Fuzzy model for mathematical problem-solving

In this section we are going to adapt our general fuzzy model in order to represent the process of mathematical problem-solving.

4.1 History

Problem–Solving (P-S) is a principal component of mathematics education with a long history and has supported numerous research programs at all levels. Given the importance of P-S, the orientations and structure of many curriculum proposals and teaching models throughout the world have been either directly or indirectly influenced by it.

In earlier papers [53, 56], we have examined the role of P-S in learning mathematics and we have attempted a review of the progress of research on P-S in mathematics education from the time that Polya presented his first ideas on the subject until nowadays. Here is a rough chronology of that progress:

1950’s – 1960’s: Polya’s ideas on the use of heuristic strategies in P-S [29-32].
1970’s: Emergency of mathematics education as a self – sufficient science (research methods were almost exclusively statistical). Research on P-S was mainly based on Polya’s ideas.
1980’s: A framework describing the P-S process, and reasons for success or failure in P-S, e.g. see Schoenfeld [36, 38], Lester, Garofalo & Kroll [24], etc.
1990’s: Models of teaching using P-S, e.g. constructivist view of learning (see [55] and its references), Mathematical modelling and applications (see [52] and its references), etc.
2000’s: While early work on P-S focused mainly on analyzing the P-S process and on describing the proper heuristic strategies to be used in each of its stages, more recent investigations have focused mainly on solvers’ behaviour and required attributes during the P-S process; e. g. MPS Framework of Carlson and Bloom [4], Schoenfeld’s theory of goal-directed behaviour [39], etc.

4.2 The Multidimensional Problem Solving Framework of Carlson and Bloom

Carlson and Bloom [4] drawing from the large amount of literature related to P-S developed a broad taxonomy to characterize major P-S attributes that have been identifying as relevant to P-S success. This taxonomy gave genesis to their
Multidimensional Problem-Solving Framework (MPSF), which includes four phases: Orientation, Planning, Executing and Checking. It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated throughout the remainder of the solution process; only in a few cases a solver obtained linearly the solution of a problem (i.e. he/she made this cycle only once). Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases, that is planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning phase of the P-S process, the solvers were at times engaged in a conjecture-imagine-evaluate (accept/reject) sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning [4; Figure 1].

It is of worth to notice that there are many similarities among the five stages of Schoenfeld’s expert performance model [36] and the four phases of MPSF. In fact, the stage of analysis of the problem of Shoenfeld’s model corresponds to the phase of orientation, the stage of design corresponds to the phase of planning, the stage of exploration corresponds to the conjecture-imagine-evaluate sub-cycle connected to the phase of planning, the implementation of the solution corresponds to the phase of executing and finally the stage of verification corresponds to the phase of checking. The qualitative difference between these two models is actually that, while the former focuses on describing the P-S process and the proper heuristic strategies to be used in each of its stages, the latter focuses on solver’s behaviour and required attributes during the P-S process [56; section 4].

4.3 The model

The construction of our fuzzy model for the P-S process is based on MPSF. For this, we consider a group of n students, n ≥ 2, in classroom during the P-S process and we denote by S_i, i=1, 2, 3 the phases of planning, executing and checking. To each of the S_i’s we attach a fuzzy subset, say A_i, of U by defining the membership function m_{A_i} exactly as in section 2. The phase of orientation being a preliminary step of the P-S process remains out of our fuzzy representation in terms of building up the assumed real system. Alternatively, it could be considered as a sub-phase of planning. The development of the rest of the model relies upon the lines of our general model presented in section 2.

4.4 Classroom applications

The following two experiments performed recently at the Graduate Technological Educational Institute (T.E.I.) of Patras in Greece. In the first of them our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems (see Appendix) given to students for solution (time allowed 3 hours). Before starting the experiment we gave the proper instructions to students emphasizing among the others that we are interested for all their efforts (successful or not) during the P-S process, and therefore they must keep records on their papers for all of them, at all stages of the P-S process. This manipulation enabled as in obtaining realistic data from our experiment for each stage of the P-S process and not only those based on students’ final results that could be obtained in the usual way of graduating their papers.
Our characterizations of students’ performance at each stage of the P-S process involved:

- Negligible success, if they obtained (at the particular stage) positive results for less than 2 problems.
- Low success, if they obtained positive results for 2, 3, or 4 problems.
- Intermediate success, if they obtained positive results for 5, 6, or 7 problems.
- High success, if they obtained positive results for 8, or 9 problems.
- Complete success, if they obtained positive results for all problems.

Examining students’ papers we found that 15, 12 and 8 students had intermediate, high and complete success respectively at stage of planning. Therefore we obtained

\[ n_{1a}=n_{1b}=0, \quad n_{1c}=15, \quad n_{1d}=12 \quad \text{and} \quad n_{1e}=8. \]

Thus, by the definition of \( m_{A_i}(x) \), planning corresponds to a fuzzy subset of \( U \) of the form:

\[ A_1 = \{(a,0),(b,0),(c,0.5),(d,0.25),(e,0.25)\}. \]

In the same way we represented the stages of executing and checking as fuzzy sets in \( U \) by

\[ A_2 = \{(a,0),(b,0),(c,0.5),(d,0.25),(e,0)\} \]

and

\[ A_3 = \{(a,0.25),(b,0.25),(c,0.25),(d,0),(e,0)\} \]

respectively.

Using the definition given in section 2 we calculated the membership degrees of the \( S^3 \) (ordered samples with replacement of 3 objects taken from 5) in total possible students’ profiles (see column of \( m_s(1) \) in Table 1). For example, for \( s=(c, c, a) \) one finds that

\[ m_s = m_{A_1}(c). m_{A_2}(c). m_{A_3}(a) = (0.5)(0.5)(0.25) = 0.06225. \]

It turned out that \( (c, c, a) \) was one of the profiles of maximal membership degree and therefore the possibility of each \( s \) in \( U^3 \) is given by \( r_s = \frac{m_s}{0.06225} \).

Calculating the possibilities of all profiles (see column of \( r_s(1) \) in Table 1) one finds that the ordered possibility distribution for the student group is:

\[ r_1=r_2=1, r_3=r_4=r_5=r_6=r_7=r_8=0.5, r_9=r_{10}=r_{11}=r_{12}=r_{13}=r_{14}=0.258, \]

\[ r_{15}=r_{16}= \ldots \ldots = r_{125}=0. \]

Thus with the help of a calculator we found that

\[ \text{ST}(r) = \frac{1}{\log 2} \left[ \sum_{i=2}^{14} (r_i - r_{i+1}) \log \frac{i}{\sum_{j=1}^{i} r_j} \right] \]

\[ \approx \frac{1}{0.301} \left[ 0.5 \log \frac{2}{2} + 0.242 \log \frac{8}{5} + 0.258 \log \frac{14}{6.548} \right] \]

\[ \approx 3.32[(0.242)(0.204)+(0.258)(0.33)] \approx 0.445 \]
\[ N(r) = \frac{1}{\log 2} \left[ \sum_{i=1}^{n} (r_i - r_{ii}) \log i \right] = \frac{1}{\log 2} [0.5 \log 2 + 0.242 \log 8 + 0.258 \log 14] \]

\[ \approx 0.5 + 3 \cdot (0.242) + (0.857) \cdot 1.146 \approx 2.208 \]. Therefore we finally obtained that \( T(r) \approx 2.653 \).

| Table 1: Profiles with non zero pseudo-frequencies |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|     |     |     |     |     |     |
| \( A_1 \) | \( A_2 \) | \( A_3 \) | \( m_s(1) \) | \( r_s(1) \) | \( m_s(2) \) | \( r_s(2) \) | \( f(s) \) | \( r(s) \) |
| b     | b     | b     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| b     | b     | a     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| b     | a     | a     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| c     | c     | c     | 0.062 | 1     | 0.062 | 1     | 0.124 | 1     |
| c     | c     | a     | 0.062 | 1     | 0.062 | 1     | 0.124 | 1     |
| c     | c     | b     | 0     | 0     | 0.031 | 0.5   | 0.031 | 0.25  |
| c     | a     | a     | 0     | 0     | 0.031 | 0.5   | 0.031 | 0.25  |
| c     | b     | a     | 0     | 0     | 0.031 | 0.5   | 0.031 | 0.25  |
| c     | b     | b     | 0     | 0     | 0.031 | 0.5   | 0.031 | 0.25  |
| d     | d     | a     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |
| d     | d     | b     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |
| d     | d     | c     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |
| d     | a     | a     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| d     | b     | a     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| d     | b     | b     | 0     | 0     | 0.016 | 0.258 | 0.016 | 0.129 |
| d     | c     | a     | 0.031 | 0.5   | 0.031 | 0.5   | 0.062 | 0.5   |
| d     | c     | b     | 0.031 | 0.5   | 0.031 | 0.5   | 0.062 | 0.5   |
| d     | c     | c     | 0.031 | 0.5   | 0.031 | 0.5   | 0.062 | 0.5   |
| e     | c     | a     | 0.031 | 0.5   | 0     | 0     | 0.031 | 0.25  |
| e     | c     | b     | 0.031 | 0.5   | 0     | 0     | 0.031 | 0.25  |
| e     | c     | c     | 0.031 | 0.5   | 0     | 0     | 0.031 | 0.25  |
| e     | d     | a     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |
| e     | d     | b     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |
| e     | d     | c     | 0.016 | 0.258 | 0     | 0     | 0.016 | 0.129 |

(The outcomes of the table are written with accuracy up to the third decimal point)

A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. Working as above we found that

\[ A_1 = \{(a, 0), (b, 0.25), (c, 0.5), (d, 0.25), (e, 0)\}, \]

\[ A_2 = \{(a, 0.25), (b, 0.25), (c, 0.5), (d, 0), (e, 0)\} \quad \text{and} \]

\[ A_3 = \{(a, 0.25), (b, 0.25), (c, 0.25), (d, 0), (e, 0)\}. \]

Then we calculated the membership degrees of all possible profiles of the student group (see column of \( m_s (2) \) in Table 1). It turned out that the maximal membership degree was again 0.06225, therefore the possibility of each \( s \) is given by the same formula as for the first group. Calculating the possibilities of all profiles (see column
of $r_s(2)$ in Table 1) we found that the ordered possibility distribution of the second group is:

$$r_1=r_2=1, \ r_3=r_4=r_5=r_6=r_7=r_8=0.5, \ r_9=r_{10}=r_{11}=r_{12}=r_{13}=0.258, \ r_{14}=r_{15}=\ldots=0$$

Finally, working in the same way as above we found that $T(r) = 0.432 + 2.179 = 2.611$. Therefore, since $2.611 < 2.653$, it turns out that the second group had in general a slightly better performance than the first one.

Next, in order to study the combined results of behaviours of the two groups, we introduced the fuzzy variables $A_i(t)$, $i=1, 2, 3$ and $t=1, 2$, as we have described in the previous section. Then the pseudo-frequency of each student profile $s$ is given by $f(s) = m_s(1) + m_s(2)$ (see corresponding column in Table 1). It turns out that the highest pseudo-frequency is 0.124 and therefore the possibility of each student’s profile is given by $r(s) = \frac{f(s)}{0.124}$. The possibilities of all profiles having non-zero pseudo-frequencies are presented in the last column of Table 1.

5 Discussion and conclusions

In this paper we developed a general model for representing several processes in Mathematics Education involving fuzziness and uncertainty. To each of the main stages of these processes we attached a fuzzy subset of the set of linguistic labels of negligible, low intermediate, high and complete success respectively by students at this stage and we used the total possibilistic uncertainty as a measure of students’ capacities.

Applications were sketched of the above model (or similar ones) attempted in earlier papers for an effective description of situations involving fuzziness and uncertainty, mainly in the area of Mathematics Education (learning, mathematical modelling), but also in the areas of Artificial Intelligence (case-based reasoning) and Management. Our fuzzy models, apart from quantitative information, they also give a realistic qualitative view of the process that they represent through the study of all possible profiles of the subjects involved during the process. Another advantage of them is that they give the opportunity for a combined study of results of two or more groups (or systems) during the same situation, or alternatively for a combined study of results of the same group (or system) during two or more different situations.

We also adapted properly our general model in order to succeed a fuzzy representation of the mathematical P-S process and we presented two classroom experiments illustrating our results in practice. Nevertheless further research is needed for the P-S process. In fact, as a general conclusion of all findings from research studies on P-S it turns out that success in P-S appears to stem from the ability to draw on a large reservoir of well-connected knowledge, heuristics and facts, from the ability to manage the emotional responses, as well as from an adequate degree of practice [53, 56]. However, although many studies have investigated and compared the characteristics of novice and expert problem solvers [23, 37, 44, etc], many of the qualitative differences appearing among them still do not seem to be completely understood. It is hoped therefore that the use of our fuzzy model as a tool in future research on P-S could lead to practical ways of restoring the weaknesses appearing to novices with respect to the expert problem solvers. It is also expected to be able in
future to extend our general model in representing further situations involving fuzziness and uncertainty in Education in general and/or in other scientific areas. Notice that analogous efforts to use principles of fuzzy logic in Education have been attempted in past by other researchers as well ([10], [25], [27], [42], [45], etc).

We must finally underline the importance of use of stochastic methods (Markov chain models) as an alternative approach for the same purposes; e.g. [47, 48, 50, 52, 58, 60, 62, etc]. Nevertheless Markov models, although easier sometimes to be applied in practice by a non expert (e.g. the teacher), apart from the quantitative information that they provide - e.g. measures for the P-S, or model-building abilities of student groups, short and long-run forecasts (probabilities) for the evolution of various phenomena, etc- they are self restricted in describing the ideal behaviour only of the subjects involved in the situation that they represent. Therefore one could claim that the fuzzy models presented in this paper are more useful to the researcher for a deeper study of the corresponding real situation, because, apart from the quantitative information, they provide also the possibility of a realistic qualitative analysis of the problems involved.

References


APPENDIX: List of the problems given to students for solution in our classroom experiments

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how can we run the maximum possible quantity of the water?

Problem 2: Given the matrix
\[
\begin{pmatrix}
1 & 2 & 2 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
and a positive integer n, find the matrix \(A^n\).

Problem 3: Calculate the integral \(\int \frac{x}{x^2 + 4} dx\).

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix \(\begin{pmatrix}19 & 15 \\ 13 & 5 \end{pmatrix}\) corresponds to the word SOME. Using the matrix \(E=\begin{pmatrix}8 & 5 \\ 11 & 7 \end{pmatrix}\) as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message?

Problem 5: The demand function \(P(Q_d)=25-Q_d^2\) represents the different prices that consumers willing to pay for different quantities \(Q_d\) of a good. On the other hand the supply function \(P(Q_s)=2Q_s+1\) represents the prices at which different quantities \(Q_s\) of the same good will be supplied. If the market’s equilibrium occurs at \((Q_0, P_0)\), the producers who would supply at lower price than \(P_0\) benefit. Find the total gain to producers.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before
the next lottery. Find the probability of getting all the balls that he draws out of the box different.

*Problem 7:* A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

*Problem 8:* The rate of increase of the population of a country is analogous to the number of its inhabitants. If the population is doubled in 50 years, in how many years it will be tripled? (ANSWER: In $\frac{\ln 3}{\ln 2} \approx 79$ years).

*Problem 9:* A company circulates for first time in market a new product, say K. Market’s research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week.

i) Find the market’s share for K two weeks after its first circulation, provided that the market’s conditions remain unchanged.

ii) Find the market’s share for K in the long run, i.e. when the consumers’ preferences will be stabilized.

*Problem 10:* Among all cylinders having a total surface of $180\pi \text{ m}^2$, which one has the maximal volume?